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Chern-Simons theory of quantum Hall effect

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Abstract The Chern-Simons (CS) theory of Fractional Quantum Hall Effect is reviewed here. We shall show that the CS action is generic to two dimensional systems in an external magnetic field, and proceed to demonstrate how the CS action can embody the composite fermion picture naturally, and also allow a study of Quantum Hall systems in an effective field theory. By including the spin as a dynamical degree of freedom, and a consequent determination of the response functions, the wave functions and the topological excitations, it will be shown how one can identify experiments/observables using which the veracity of the composite fermion model can be settled unambiguously

Keywords : Landau levels, Chern-Simons action, composite fermions

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1. Introduction

The purpose of this review is to discuss briefly the Chern-Simons (CS) approach to study Quantum Hall systems (QHS), with the spin degree of freedom included. While the CS terms appear in the Integer Quantum Hall effect (IQHE) as well as the 1-loop level, they are more intrinsic to the fractional quantum Hall effect (FQHE), and are introduced to incorporate the effect of electron correlations, *P, T* violations, and the composite fermion (CF) picture all together. They also reflect the topological nature of QHE, a sophisticated analysis of which (not done here !) leads to a deep understanding of the conformal nature of the edge currents and the chiral anomaly in macroscopic two dimensional condensed matter systems. Here, I shall restrict myself to discussing the implementation of the CF picture in a CS Lagrangian, without freezing the spin degree of freedom, and studying various response functions. As we shall see, this approach provides us with a means of identifying experiments that can *independently* determine all the parameters that enter into the CF model. I shall also study the wave functions that get determined in this model and compare them with the more detailed many body computations. Finally, I shall dwell upon the very interesting skyrmionic excitations and their properties that the QHS are expected to possess.

Admittedly, it is not yet completely known how exactly the CF emerge as the relevant quasi particles in QHS, from a more basic many body study. For the same reason, a 'derivation' of the CS lagrangian as an effective interaction term for the otherwise free electrons is also lacking. However, as the reader will see below, there does exist an *a posteriori* justification which follows from its successes.

It is virtually impossible to cover all the aspects of the CS theory and of CF in a single review. Fortunately, there are excellent review articles that discuss various aspects of QHE. Prominent among them are the collection of review articles spread over Refs [1-5]. There is a book exclusively devoted to a discussion of IQHE [6] and finally, a collection of articles on CF [7] is also expected to be published soon. As mentioned above, our purpose here is to concentrate on the CS theory which is as yet being developed. I hope to provide an introduction to the subject, and a setting to appreciate the current prospects/problems in the area. In that sense, the choice of the subjects dealt with here reflect a view point which the reader will surely not miss as she proceeds along.

In the next section, I briefly present the salient features of IQHE where I will also discuss the Landau level problem. After performing a 1-loop computation (RPA) to determine the response functions, I go on to Section 3 where I introduce the CS action in its general setting. Section 4 is devoted to FQHE and in Section 5, I introduce the CF model (CFM). In Section 6, central to our studies here, I elaborate upon a CS theory of QHS with multicomponents. Although I study only single layered systems with spin degree of freedom, its adaptation to multilayered systems is straight forward. Sections 7 and 8 discuss the experimental verification of CFM, in the light of CS theories, with the former section paying attention to the filling fraction $\nu = 1/2$, and the latter away from that value. I then go on to Section 9 to discuss the very interesting skyrmionic excitations, which as we shall see allow us to determine a topological parameter introduced in the CFM. In Section 10, I briefly (so brief as to be apologetic about it) mention some recent developments in our understanding the CS theory. Finally, I conclude the article in section 11 with a summary and an acknowledgment.

2. Quantum Hall effects

There are two QH effects, the integer (IQHE) and the fractional (FQHE). While the electron interaction has little role in the former, the latter owes its existence to strong electron-electron correlations. Both the effects occur in "dynamical" two dimensional ($D = 2$) systems, require disorder for their sustenance, and are seen in very high magnetic fields ($B \sim 1 - 30$ Tesla) and at low temperatures ($T \leq 1$K).

The term "dynamical" merits some elaboration here. As such, a two dimensional system is an idealization, where the third degree is frozen as a constraint. In reality, what happens is that the electrons are free to move in a plane, with the energy in the normal direction getting quantized. If the temperatures are much less than the gap in the third direction, the system may be effectively considered to be two dimensional. Note however that this by itself does not reduce the dynamics to $D = 2$ in all its entirety. For instance, the electric field produced by a charged particle continues to obey the inverse square law; electrodynamics in $D = 2$ leads to a $\frac{1}{r}$ law instead. Thus care must be exercised in modeling these systems.

In practice, a system that is planar is realized experimentally in inversion layers in semiconductors, formed at a semiconductor – insulator interface (Si – SiO_2) or in a semiconductor heterojunction such as GaAs – $Ga_xAl_{1-x}As$. When the conduction electrons occupy the lowest subband, they get trapped at the interface, but are otherwise free to move in the plane, with very long mean free paths $\sim 10^5 Å$. The band gap in the perpendicular direction is ~ meV. Thus, the system may be treated to be effectively two dimensional only at very low temperatures ($T \leq 5K$). This system is called the two dimensional electron gas (2DEG).

The QHE is exhibited by a 2DEG in a strong magnetic field perpendicular to the plane. The fields achieved in the laboratory range from ~ $10T - 30T$. If the sample has high mobility, but is not excessively clean, under suitable conditions and at very low temperatures, the Hall conductivity $\sigma_H \equiv \sigma_{xy}$ exhibits a step function like behaviour as a function of the applied field. Klitzing *et. al.* [8] found that around T ~ $1K$, σ_H was seen to obey the simple formula given by,

$$\sigma_H^N = N\sigma_0 \ ; \quad \sigma_0 = e^2/h \ ; \tag{1}$$

where N is an integer. Correspondingly, the diagonal resistivity remains zero except at the transition from one plateau to the other where it shows a sharp rise. The accuracy of the quantization found by Klitzing *et. al.* [8] was indeed high, being ~ 10 ppm. It has been subsequently found that by lowering the temperature further, the accuracy can be improved upon by another two orders of magnitude. The flatness of the plateaus also increased correspondingly, suggesting strongly that the quantization and the plateau formation with discontinuous jumps would be exact at $T = 0$. This effect has been called the Integer Quantum Hall Effect (IQHE).

It should be noted that the expression for σ_H involves only the fundamental constants, the speed of light and the fine structure constant. Given the accuracy mentioned above, it is not surprising that the QHS have already replaced the old standard of resistance and opened up new vistas in the field of metronomy [9]. Let us express the fine structure constant α as

$$\alpha = \frac{\mu_0 e^2 c}{2h} \quad \frac{\mu_0 c}{2N}\sigma_H(N), \tag{2}$$

where N is an integer. If we assume that the value of the vacuum permittivity μ_0 is known accurately (the value is $4\pi \times 10^7\ H/m$) along with that of $c(\equiv 299792458\ m/s)$, the fine structure constant gets determined with a precision limited by that of the value of σ_H ! From the measurement of the anomalous magnetic moment of the electron, $\mu = \mu_0\ (1 + \frac{\alpha}{2\pi})$, the value of α is known to an accuracy given by $\alpha^{-1} = 137.035993(10) \pm 0.07$ ppm. The current uncertainty in σ_H is 0.1ppm, which is close to the precision with which the fine structure constant is measured in atomic and particle physics experiments.

The following features are worth noting here. First of all, we have the remarkable accuracy of quantization mentioned above. This quantization is seen in fairly involved many body systems, with impurity added. The value of σ_H is independent of the details of the system : the carrier mass, the carrier magneic moment, the geometry of the sample and also the homogeneity of the magnetic field itself. It is also independent of the details of the electron-electron interaction as well as the nature of the impurity, the latter at least at $T = 0$. Thus QHE is universal and also topological, especially since it is independent of the geometry of the sample.

It is easily seen that there are two issues that are central here. The formation of the plateau, and the quantization of the plateau value. Of course, one also has to understand how the almost discontinuous transition takes place from one value of σ_{xy} to the other. Of these, we will be exclusively interested in the quantization of the plateau value, rather than the plateau formation itself. In other words, we study the system only at the centre of the plateau. I refer the reader to Refs. [1–3, 5–7) which discuss how the disorder leads to the stabilization of the quantized value of σ_H at and around each integer filling fraction.

The clue to this remarkable phenomenon of quantization is in the fact that the energy levels of the electrons in a magnetic field are quantized. If the electrons are non-interacting, or possess a weak interaction, then the energy gap is provided by the cyclotron frequency. IQHE can be understood in terms of this value for the gap. Although it might appear that the physics of the FQHE would be different because it occurs when only a fraction of the lowest Landau level (LLL) is occupied, the composite fermion picture describes FQHE to be an IQHE at some effective magnetic field, coming from the combination of the external with an induced field. So let me review the dynamics of the Landau level problem, as is appropriate in the context.

A. The Landau level problem

Let a system of non-interacting electrons be confined to say, the $x-y$ plane. Let a magnetic field of strength B be applied in a direction perpendicular to the plane ; $\boldsymbol{B} = B\hat{k}$. The Hamiltonian for the system is simply the kinetic energy

$$H = \frac{\pi_x^2 + \pi_y^2}{2m}, \tag{3}$$

where however, the momentum operators satisfy the commutation relation

$$[\pi_x, \pi_y] = \frac{ie\hbar}{c} B \equiv i\hbar^{eff}, \tag{4}$$

which implies a corresponding commutation relation for the components of the velocity operators $v_i = \pi_i / m$. It is now easily recognized that the 'free' Hamiltonian in eq. (3) is, in fact, an oscillator and has the standard spectrum $E_n = (n+\frac{1}{2})\hbar\omega_c$ where ω_c is the usual cyclotron frequency eB/mc. To complete the dynamical description, we need to make use of the full algebra of the observables in phase space. They are exhausted by

$$[x_i, x_j] = 0\,;\ [x_i, \pi_j] = i\hbar\delta_{i,j}\ ;\ [\pi_i,\ \pi_j] = i\hbar^{eff}. \tag{5}$$

The characterization of the state is now straight forward. Indeed, the classical solutions for the problem at hand, read

$$x(t) = x_0 - \frac{c}{eB}\pi_y(t);\quad y(t) = y_0 + \frac{c}{eB}\pi_x(t), \tag{6}$$

from which we can also deduce the commutation relations obeyed by the coordinates of the centre of the classical orbit :

$$[x_0, y_0] = -\frac{2c}{eB} i\hbar \ \equiv -\frac{2}{\omega_c m} i\hbar \tag{7}$$

which again exhibits the impossibility of determining the coordinates simultaneously. It may easily be verified that $[x_0, H] = [y_0, H] = 0$. It is then convenient to consider the distance operator for the centre, $R_0^2 = x_0^2 + y_0^2$ which also commutes with the Hamiltonian. Recognizing that the spectrum of R^2 is given by $R_0^{(p)2} = (p+1/2)\frac{2\hbar}{m\omega_c}$, we may construct a basis from the simultaneous eigenstates of H, R_0^2. Thus,

$$H \mid n,p> = (n+1/2)\hbar\omega_c \qquad R_0^2 \mid n,p> = (p+1/2)\frac{2\hbar}{m\omega_c} \tag{8}$$

The degeneracy ρ_l per unit area is simply the number of allowed values of R_0^2 in the unit circle; it may be read off as $\frac{eB}{2\pi c\hbar}$, which may be written in terms of the unit flux Φ_0 as Φ/Φ_0.

It is worthwhile noting that the treatment has involved only observable operators, and hence does not involve the gauge potential anywhere. The algebra listed in eq. (5) does in fact, allow us to determine any matrix element of any observable without having to fix the gauge. It is only when an explicit form of the wave function is desired does one need to introduce a gauge, in order to realize the algebra in some representation. More importantly for our purposes here, the determination of the current-current correlators involves only gauge invariant operators.

The linear response of this system to an external electric field E is also easily determined: Let $E = E_0\hat{n}$ be applied in some direction the plane. There then exists a frame of reference, obtained by a boosting, where E can be transformed away. The velocity is given by $V = \frac{E_0}{cB_0}\hat{z}\times\hat{n}$. In the new frame, the transformed magnetic field is still in the same direction, but has a different magnitude given by $B' = B + \mathcal{O}(E_0^2)$. Thus, if the electric field is infinitesmal, the second order change that B suffers may be ignored. The boost has thus restored the situation prior to the application of the E field ; and therefore, in the transformed frame, $<j> \equiv 0$, for the current. An inverse boost gives a velocity $-V$ to every electron, leading to a current density $<j> = -\rho V$ in terms of the electron concentration in the plane. The above expression may be rewritten as

$$J_i = \frac{\rho}{cB}\varepsilon^{ij}E_j = \frac{\nu\rho_l}{cB}\varepsilon^{ij}E_j = \frac{e^2\nu}{h}\varepsilon^{ij}E_j\,, \tag{9}$$

where I have expressed the electron concentration in terms of the degeneracy factor ρ_l and the filling fraction ν. As expected, the Hall conductivity shows a linear dependence on the filling fraction. In particular, at $\nu = N$, it gives the quantized values seen in the experiments. We shall not get into a discussion of how the central value gets stabilized to form a plateau in this review. Instead, eq. (9) will be rederived in a different language, by determining the quantum fluctuations at the one loop level, using the lagrangian formalism. Laborious that it might appear to be, it has the germs of generalization to a study of FQHE, which is of main interest here.

B. The one loop effects :

I shall continue to ignore the spin degree, assuming a large value for the gyromagnetic ratio g. In this limit, for the spinless fermions in an external magnetic field, the lagrangian density is given by

$$L = \psi^* iD_0\psi - \frac{1}{2m^*}|D_k\psi|^2 + \psi^*\mu\psi - eA_0^{in}\rho + \frac{1}{2}\int d^3x'\, A_0^{in}(x)\,V^{-1}(x-x')\,A_0^{in}(x'), \tag{10}$$

where the covariant derivatives are given by $D_0 = \partial_t - ie\,(A_0 + A_0^{in})$; $D_i = \partial_i - ieA_i$, in terms of the external gauge potentials A_0, A_i, and the internal scalar potential A_0^{in} . The chemical potential μ fixes the electron concentration, and m^*, ρ are the effective mass and the (mean) density respectively. The fourth term in the above equation ensures the charge neutrality of the system, and in the last term, $V^{-1}\,(x - x')$ represents the instantaneous charge interaction potential in the operator sense. Indeed, the last term is merely the usual interaction term with quartic contribution from the fermion fields, as it should for a density-density case. The form of V is either the usual $1/r$ or some other short range potential. In other words, the internal dynamics is governed by the (3+1) dimensional Maxwell Lagrangian, as is appropriate for the medium.

The partition function

$$\mathcal{Z}\,[A_\mu] = \int\, [dA_0^{in}][d\psi][d\psi^*]\, e^{i\int d^3x\, \mathcal{L}(x)}, \tag{11}$$

is to be evaluated from the Lagrangian density in eq. (10). Note that I am employing a compact four vector notation to denote the gauge potentials, although no Lorentz invariance is implied. Integrating over the fermionic fields first, the resulting partition function can then be written in terms of the effective action

$$S = -i\,Tr\ln\left[iD_0 + \mu + \frac{1}{2m^*}D_\kappa^2\right] - e\int d^3x\, A_0^{in}(x)\rho + \frac{1}{2}\int d^3x \int d^3x'\, A_0^{in}(x)V^{-1}(x-x')A_0^{in}(x') \tag{12}$$

for the gauge fields, incorporating the bulk effect of fermions of the system.

Let us now switch on the quantum fluctuations of the gauge fields, about the background value. Keeping only the terms quadratic in fluctuations, the effective action in terms of the fluctuations may be written as

$$S^{(1)} = -\frac{1}{2}\int d^3x \int d^3x'\, A_\mu(x) \prod{}^{\mu\nu}(x,x')\, A_\nu(x'), \tag{13}$$

where the polarization tensors

$$\prod{}^{\mu\nu}(x,x') = \left.\frac{\delta^2 S}{\delta A_\mu \delta A_\nu}\right|_{\langle A_0^{in}\rangle}, \tag{14}$$

$$= -i\langle j^\mu(x) j^\nu(x')\rangle_c - i\left\langle \frac{\delta j^\mu(x)}{\delta A_\nu(x')}\right\rangle_c . \tag{15}$$

Here $\langle \ldots \rangle$ represents the expectation value in the background field ground state, and C refers to the connected diagrams of the two particle correlation functions. $\mathcal{A}_\mu$ is the fluctuating field, and $j^\mu(x)$ is the fermionic current operator.

Note that the polarization tensor satisfies, by virtue of gauge invariance,

$$\partial_\mu^x \prod{}^{\mu\nu}(x,x') = 0. \tag{16}$$

It is instructive to resolve the tensor in a basis which incorporates the gauge, the translational and the rotational degrees of freedom. In the Fourier space, the resolution is given by

$$\Pi^{\mu\nu}(\omega, \boldsymbol{q}) = \Pi_0(\omega, \boldsymbol{q})(q^2 g^{\mu\nu} - q^\mu q^\nu) + (\Pi_2(\omega, \boldsymbol{q}) - \Pi_0(\omega, \boldsymbol{q}))$$
$$\times \left(\boldsymbol{q}^2 \delta^{ij} - q^i q^j\right) \delta^{\mu i} \delta^{\nu j} + i\Pi_1(\omega, \boldsymbol{q}) \varepsilon^{\mu\nu\lambda} q\lambda, \tag{17}$$

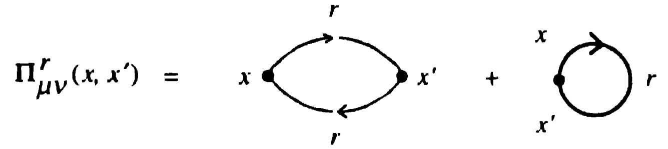

Figure 1. The polarization tensor $\Pi^r_{\mu\nu}(x, x')$ is diagrammatically shown. Here x and x' are two space-time points. In the second diagram, x and x' correspond to same space-time point. The lines with arrows represent the single particle Green's functions.

where $q^2 = \omega^2 - \boldsymbol{q}^2$. The occurrence of Π_1, the P, T, violating form factor, is to be noted. An evaluation of the above form factors in the lowest order in $\boldsymbol{q}^2$ yields

$$\Pi_0(\omega, \boldsymbol{q}) = -\left(\frac{e^2}{2\pi}\right) \frac{\omega_c}{\omega^2 - \omega_c^2} p, \tag{18}$$

$$\Pi_1(\omega, \boldsymbol{q}) = -\left(\frac{e^2}{2\pi}\right) \frac{\omega_c^2}{\omega^2 - \omega_c^2} p, \tag{19}$$

$$\Pi_2(\omega, \boldsymbol{q}) = -\frac{e^2}{4\pi m^*} \omega_c^2 \left[\frac{3}{\omega^2 - \omega_c^2} - \frac{4}{\omega^2 - 4\omega_c^2}\right] p^2 . \tag{20}$$

The linear response of the system to external electromagnetic probes is contained in the three form factors that we have determined above. Of particular significance is the occurrence of the form factor Π_1 which multiplies a tensor which violates both parity and time reversal, reflecting the dynamics in the presence of the background (magnetic) field. Indeed, it is a simple matter to derive the conductivity tensor for a static uniform electric field, and one finds that

$$\sigma_{i,j} = \Pi_1 \varepsilon_{i,j}, \tag{21}$$

in terms of the completely antisymmetric Levi-Civita (pseudo) tensor. As a consequence of the above equation, the resistivity tensor is also antisymmetric, leading to a vanishing of the diagonal resistance. Thus both the diagonal conductivity and the diagonal resistivity vanish simultaneously for these systems !

It is instructive to examine further the nature of the term in the effective action involving Π_1. The term, which is peculiar to $D = 2$ is, in fact, the Chern-Simon (CS) action. It has certain highly non-trivial features which I shall discuss in the next section.

3. The Chern-Simons action

The CS action was first introduced in the study of 2+1 dimensional systems to generate a mass for the photon in a gauge invariant fashion [10]. It has since then spawned a rich and fertile field of investigation in particle as well as condensed matter physics. For our purposes here, I shall discuss only those features that are easily seen and are also essential.

The CS lagrangian in $D = 2$ for Abelian fields is given by

$$\mathcal{L}_{CS} = \frac{\theta}{2}\varepsilon^{\mu\nu\rho}A_\mu \partial_\nu A_\rho = \frac{\theta}{4}\varepsilon^{\mu\nu\rho}F_{\mu\nu}A_\rho \,. \tag{22}$$

As mentioned above, this form is peculiar to $D = 2$, constructed as it is from the Levi-Civita tensor.

The above Lagrangian is to be compared with the standard Maxwell form $F^{\mu\nu}F_{\mu\nu}$ which is quadratic in the derivatives of the gauge fields, possessing the usual kinetic energy form. $\mathcal{L}_{CS}$ is, in contrast, a 'derivative' lagrangian, being linear in the field tensor. The resultant equation of motion leads to identities, and not to differential equations. Indeed, coupling to a matter current, let us write

$$\mathcal{L} = \mathcal{L}_{CS} + ej_\mu A^\mu . \tag{23}$$

The resultant equation of motion

$$eJ^\mu = \theta\varepsilon^{\mu\nu\rho}F_{\nu\rho} \,, \tag{24}$$

merely enslaves the charge and the current densities to the fields. Note that the roles of the electric and the magnetic fields are reversed here. The charge density produces a magnetic field, and the current density an electric field (in the transverse direction in the plane), exhibiting both parity and time reversal violations. The classical dynamics is of course trivial. And indeed, the energy-momentum carried by the fields is also identically zero, as can be verified by taking the functional derivative of the action w.r.t. the metric tensor $g_{\mu\nu}$.

Is the quantum dynamics also trivial ? Appearances apart, it is non-trivial, and has a rich structure not available in conventional actions. Notice that the field produced by a point particle is a delta function located at the position of the particle. The interaction between two hard core particles is then governed not by the Lorentz force, but by the Aharanov-Bohm phase, which depends on the value of the coupling constant θ. The transition, force $\rightarrow$ phase, for a many particle system has some surprising consequences. The N particle Hilbert space $\mathcal{H}_N$ cannot be written as a direct sum of single particle Hilbert spaces ; the group associated with the exchange of particles is not the permutation group, but the more complicated Braid group The statistics associated with the exchange of the particles is also not just fermionic or bosonic. the particles can pick up any phase on exchange, and are called anyons [11]. The statistical mechanics of N non-interacting anyons is so involved that analytic results exist only up to the second virial coefficient [13]. And most importantly, being defined in terms of the Levi-Civita, rather than the metric tensor, the CS action is also topological.

One final comment. The CS action derived above at the 1-loop level is also exact, thanks to a theorem proved by Coleman and Hill [12] which guarantees that higher order quantum corrections for the CS term do not exist. Thus the Hall-conductivity determined above is also exact.

The 2DEG has thus provided an explicit realization of the above action in Quantum Hall systems. We shall explore some of the aspects of its manifestation in fractional quantum Hall systems below. At this juncture it is good to remember that the occurrence of CS action is generic to 2DEG in an external magnetic field. While the details of its manifestation may depend on the details of the system, its occurrence itself is perhaps unavoidable.

4. Fractional quantum Hall effect

Let me now consider the dynamics of 2DEG when the filling fraction $n < 1$. We are essentially in the large B limit, and in this case, there is *a priori* no reason to see any further plateau formation, especially as there is no gap in the lowest Landau subband itself. However, in their surprising discovery, Tsui *et. al.* [14] observed a further occurrence of a series of plateaus at fractional filling fractions. This was seen in samples of much higher mobility, was destroyed easily by disorder, and at much lower temperature. The precision of quantization which takes place at rational filling fractions is quite high, being ~ 1 part in 10^5. Clearly, since the magnetic field has already served to confine all the electrons in the lowest subband, the occurrence of the gap – so essential for the plateau formation – would be due to the electron-electron interaction, $V_{int}(\vec{r}_i - \vec{r}_j)$ which is to be measured in the scale $e^2 / \epsilon_0 \ell$ expressed in terms of the dielectric constant of the medium and the magnetic length ℓ.

We are thus saddled with the burden of invoking the (possibly screened) Coulomb interaction between the electrons in order to produce a gap, and of course the right Hall conductivity at the right filling fractions. Complicated that it might appear at first sight, there exists a simple proposal to encapsulate the effect of the correlations on the electrons by simply attaching flux tubes to the electrons. The new particles so obtained are therefore composites of flux and charge, and are called composite fermions (CF). The crux of this beautiful idea, due to Jain [15], is that the composite fermions are free or, if at all, weakly interacting. As the name suggests, the attachment of the flux tubes does not alter the statistics of the particles. Thus, it is always an even number of flux quanta that get attached. Jain proposed that these quasi particles are the relevant degrees of freedom (in the Fock space) in order to discuss FQHE.

Let me elaborate on the idea of CF further. Consider first the tying up of the charge density to the magnetic flux. Such a thing is possible, as we observed, only if P, T are both violated. Granting that it is natural in the light of our findings in the last section, it is still not of much use as the dynamics is too involved for a rigorous analysis. Here comes the next assumption in the CF approach : that one can do a mean field (MF) analysis. If we write $\rho(x) = \bar{\rho}(x) + \delta\rho(x)$ the fluctuation about the mean field is treated as a (small) perturbation. Indeed, there then follows a corresponding smearing for the singular flux tubes that get attached to the electrons. The resultant magnetic field will not be singular, but be simply proportional to $\bar{\rho}(x)$. If one takes the mean density to be uniform, it follows that there is also a mean internal magnetic field which is also uniform. In the MF approximation, the composite fermions see an effective field which is the sum of the external and the induced internal field. Jain [15] suggested that an IQHE with the effective field translates into FQHE with the original field.

We thus see that the CF model (CFM) has two ingredients. The existence of CF as quasi particles, and the MF ansatz. And both need to be justified. Consider the latter ; a simple assumption that it is, it also involves a major simplification in the dynamics. The singular flux tubes can only give phases which are non-classical. The smearing on the other hand converts the phases into ordinary forces. It is not at all clear how this comes about. However, it turns out that the assumption is not unrealistic, as it will be seen below. Hereafter, CFM will be meant to mean both the assumptions mentioned above.

5. Composite Fermion model

We consider the details of the model now. Clearly, there are two free parameters in the model, the number of flux tubes ($2s$), and the number of Landau levels (p) filled in the effective field. A simple algebra relates the true filling fraction ν to the above parameters by the relation

$$\nu = \frac{p}{2sp+1}, \tag{25}$$

which follows from $B_{eff} = B - 2sp\phi_0$, where $\phi_0 = hc/e$ is the unit flux quantum. It should be noted that the integer p is not constrained to be positive. Indeed, it is possible that the effective field may be in a direction opposite to the applied field, in which case p can be taken to be negative.

The above sequence is the first indication that CFM may be viable ; Indeed, when it was first proposed, the then known filling fractions $\nu = 2/3, 2/5$ etc easily fit into the sequence, apart from the well known Laughlin sequence $\nu = 1/(2s+1)$ the members of which had been first seen experimentally. It is interesting that all the members of the Laughlin sequence correspond to IQHE with the lowest filling fraction, $p = 1$. When no flux tube is attached, *i.e.*, $s = 0$, the Jain sequence reduces to the IQH states In that sense, the CF picture has IQH states at the lowest level in the hierarchy of flux tube attachment.

Is it possible to write the wave functions for the Jain sequence, if we know the wave function for the corresponding integer filling ? Arguing that the addition of $2s$ flux quanta create that many additional zeros in the IQH state wave function, Jain proposed that

$$\psi = \mathcal{P}\prod_{i<j}(z_i - z_j)^{2s}\Phi_p \tag{26}$$

where Φ_p is the many body wave function for a IQHS with p fully filled Landau level. Here $2s$ is the number of flux quanta attached to each particle, $z_j = x_j + iy_j$ is the complex coordinate of the j-th particle and $\mathcal{P}$ refers the projection onto the lowest Landau level. These trial wave functions have been verified numerically to be excellent approximations to the true many body states, with an overlap of more than 99%, providing the first vindication of the CF proposal. Note however that the prescription for ψ_{CF} above goes beyond the assumption of attaching flux tubes. That would cause only a change in the phase. Presumably, the flux tubes are extended, making the particles (more hard core). We will come back to this aspect later.

Laughlin had earlier argued [16] that the $\nu = 1/(2s+1)$ states should have excitations carrying a fractional charge $e^* = e/(2s+1)$. Halperin [17] subsequently showed that these excitations obey fractional statistics as well. Recent experiments involving Shottky noise [18] have verified that the excitations for $\nu = 1/3$ state have indeed a fractional charge given by $e^* = e/3$. On the other hand, the CFM predicts that the excitations do not carry any fractional charge. It might thus appear that the CFM is in conflict with the experiments. The answer to this has been provided by Halperin [19], who carefully analysed the nature of the excitations in CFM. Going beyond the MF approximation, Halperin argued that there is no conflict as such : indeed, in CFM both the charge and flux are excited. The charge excitation across the gap in the above states corresponds to the creation of a quasiparticle and a quasihole simultaneously, but at two different points. The quasiparticles carry charge e with $2s$ unit flux quanta. As a result of piercing the fluxes through the system, an amount of charge equal to $2s\nu e$ will be pushed towards the boundary. However, the charges at the boundary get cancelled by the simultaneous creation of quasiparticle and quasihole. Therefore at the point of creation of quasiparticle, the remaining charge is $e/(2sp+1)$ which is consistent with the Laughlin quasiparticle.

We now get to see glimpses of the correctness of the CFM. But we do not as yet possess a means of studying the bulk properties of the system, in a tractable manner. Notice also that the most dramatic consequence of CFM is when $p \rightarrow \infty$. In that limit, we get even denominator states, which correspond to $B_{eff} = 0$, making the system gapless. It would be therefore the right regime to test the veracity of CFM. It turns out that the right track is provided by a CS theory, which can be used to study all the desired properties of the QHS. This should not be surprising, if we bear in mind the arguments in the section 4. It is pleasing nevertheless, because a CS theory would then mean an innate universality in the FQHE as much as in IQHE. The CS model is also natural since the CF proposal essentially ties down a magnetic field to the charge density – veritably a hall-mark of the CS action. I shall devote the rest of the review for discussing QHE within the CS model.

6. Chern-Simons theory of quantum Hall systems

The basic idea is to translate the CFM into a CS language. This was first done by Lopez and Fradkin [20, 21] in the fermion picture, by adding a CS term, *at the tree level*, which attaches the requisite number of flux tubes to the electrons. They then proceeded to study the dynamics in the mean field (MF) approximation, and perturbations over the MF results. The analysis of Lopez and Fradkin [20, 21] is not any more complicated than that of IQHE, except that there is an extra gauge field coming from the CS action to integrate over when the 1-loop effective lagrangian is determined. Before embarking on this fairly straight forward programme, let me pause to discuss yet another very interesting aspect of QHS, *viz.* the possibility of spin being dynamical. I shall present a formalism that incorporates the spin in a natural manner, and a model that realises almost all the experimentally known QHS. This would allow us to study QHE in a general setting. The formalism is also close to the one used for multi layer systems, differing in details. I will emphasize the spin degree of freedom here, but merely allude to the results for multi layer systems. (See S. M. Girvin and Macdonald, and J. P. Eisenstein, in Ref. [5] for a detailed discussion of the theoretical and experimental aspects of multicomponent QHS.)

A. *Quantum Hall systems with the spin degree*

The analysis in the previous sections ignored the spin degree of freedom completely. The reason is that the Zeeman gap for the spin is normally much larger than the orbital part. The spin degree then gets frozen and has practically no dynamical role to play. This assumption fails, however, for relatively low density GaAs samples. As Halperin [19] argued, such a sample possesses very low Zeeman energy ($g \sim 0.45$) compared to cyclotron energy (recall that $M^*=0.07\, m_e$), and is less than even the Coulomb energy which is responsible for producing the charge gap in the FQHE states. This led him to conclude that all the QHS may not be fully polarized ; some of them could be even unpolarized[1]. Indeed, experiments reveal that at relatively small

[1]See Khandelwal *et al.*, *Cond-mat* 9801119, where, for the first time, the spin polarizations near $\nu = 1/3$ have been determined using optically pumped nuclear magnetic resonance studies of the Knight Shift.

values of B, the QHS at filling factors $\nu = \frac{4}{3}, \frac{8}{5}, \frac{10}{7}$ (Refs. [22, 23]) and $\frac{2}{3}$ (Refs. [23, 24]) are unpolarized (spin singlet) while the states at $\nu = \frac{3}{5}$ (Ref. [24]) and $\frac{7}{5}$ (Ref. [22]) are partially polarized. Further, it is also known experimentally that the states which are in the partially polarized or unpolarized phase to start with pass over to their fully polarized phase as the Zeeman energy is increased sufficiently — either by increasing the tilting angle of the magnetic field keeping the perpendicular component of it fixed [22-25], or by increasing the electron density [25], since the value of g becomes enhanced with the densities [29]. In the vanishing Zeeman splitting (VZS) limit, it has been found from numerical computations [26–28] that the states with $\nu = 2/(2n+1)$ (n is an integer) are unpolarized and those of the Laughlin sequence [16] with $\nu = 1/(2n+1)$ are fully polarized, in the thermodynamic limit. Also the state at $\nu = \frac{3}{5}$ has been found to be partially polarized by an exact diagonalization study [30], in agreement with the experiments. Wu *et al.* [31] have constructed trial wave functions for these states by employing the CFM and found good overlap with the numerically diagonalized states. They report that, in the VZS limit, all the even numerator QHS are unpolarized and all those states which have both the numerator and denominator (of ν) odd are partially/fully polarized.

Does CFM hold for QHS with the spin freedom included ? Unlike the orbital part where only the normal component is relevant for the planar system at hand, the spin degree has the full $SU(2)$ symmetry, as it is not constrained to be along the normal to the plane. Perhaps motivated by this, Frohlich *et al.* [32] and Balatsky and Fradkin [33] have attempted to study the unpolarized QHS, by employing a non-abelian CS model. In this picture, the electrons are composites of holons and spinons. While charged spinless holons interact with a $U(1)$ CS gauge field, neutral spin-1/2 spinons interact with a $SU(2)$ CS gauge field. Both the holons and the spinons obey semionic statistics, and combine specifically to ensure the statistics of the electron to be fermionic. Note that apart from the complicated interaction involving $SU(2)$ CS gauge field, the limitation of this model is that it can not describe all the arbitrarily polarized QHS.

The above approach is open to criticism. Apart from its lack of generality (since only the singlet states are treated), there is *per se* no reason why a non-abelian CS theory is needed Indeed, the underlying fundamental theory is the abelian Maxwell theory, and any effective theory obtained by integrating over the unwanted degrees of freedom can only be expected to be abelian. Such a purely abelian model is, in fact, quite easy to construct, and is also a natural extension of the standard CFM. Naturally, the fully polarized QHS emerge as a special case, as also the integer QHS.

The model involves a pair of abelian CS fields[2] and has been designated as the doublet model. In general, the model incorporates the idea that each particle has two kinds of vortices— one of which is seen by like spin particles and the other by the particles of unlike spin.[3] I shall discuss the model below.

B. The doublet model

It is convenient to work in the natural units $\hbar = c = 1$, which will be employed henceforth.

[2]Models with matrix valued coupling have been considered earlier, in a different context by X. G. Wen and A Zee, *Phys. Rev.* **B.46** 2290 (1992)

[3]A similar interpretation has been given by Belkhir and Jain [34] However, their interpretation does not lead to the unpolarized sequence $\nu = 2n/(3n+2)$.

To describe it briefly, the model incorporates a doublet of U(1) Chern-Simons (CS) gauge fields which couple to the fermions as follows :

$$\mathcal{L} = \psi_\sigma^\dagger [i\partial_0 - ea_0^\sigma]\psi_\sigma - \frac{1}{2m^*}\left|[\partial_j - ie(A_j + a_j^\sigma)]\psi_\sigma\right|^2 + \frac{1}{2}\varepsilon^{\mu\nu\lambda}\tilde{a}_\mu^\sigma \Theta'_{\sigma\sigma'}\partial_\nu a_\lambda^{\sigma'}$$

$$-\frac{1}{2}\int dr'\delta\rho(r)V(|r-r'|)\,\delta\rho(r') + \frac{g}{2}\mu_B\psi^\dagger\vec{\sigma}\cdot\vec{B}\psi\,. \tag{27}$$

Several features are to be recognized here. First of all, the CS term has a matrix valued coupling, and has the general form $\Theta' = \theta_1' + \sigma_1\theta_2'$, where σ_1 is the usual Pauli matrix. In the diagonalized basis, one component of the CS gauge field $a_\mu^+ = a_\mu^\uparrow + a_\mu^\downarrow$ couples to all the fermions irrespective of the spin while the other component of the CS gauge field $a_\mu^- = a_\mu^\uparrow - a_\mu^\downarrow$ distinguishes the spins. The respective strengths are given by $\theta'_+ + \theta'_1 + \theta'_2$ and $\theta'_- + \theta'_1 - \theta'_2$. It is shown in Ref. [14] that after rescaling of the CS gauge fields and their strengths, if the strengths of the CS gauge fields $a_\mu^\pm$ are chosen to be $(e^2/2\pi)$ $(1/2s)$ (s is an integer) and ∞ respectively by the requirement of composite fermions, the model describes arbitrarily polarized QHS with correct spin polarization which is given by

$$\nu = \frac{p\uparrow + p\downarrow}{2s(p\uparrow + p\downarrow)+1}, \quad \frac{\Delta\bar{\rho}}{\bar{\rho}} = \frac{p\uparrow - p\downarrow}{p\uparrow + p\downarrow}. \tag{28}$$

Here $p\uparrow$ ($p\downarrow$) is the effective number of Landau levels filled by spin up(down) composite fermions, $2s$ is the even integral number of flux quanta attached to each particle, and $\bar{\rho}$ and $\Delta\bar{\rho}$ are the mean density and spin density respectively. We, therefore, essentially have only one CS gauge field a_μ^+ while a_μ^- decouples. We thus write the Lagrangian density (dropping subscript and superscript of the CS gauge field and its strength),

$$\mathcal{L}[\psi, a_\mu] = \psi^\dagger[i\partial_0 - ea_0]\psi - \frac{1}{2m^*}\left|[\partial_j - ie(A_j + a_j)]\psi\right|^2 + \frac{\theta}{2}\varepsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda$$

$$-\frac{1}{2}\int dr'\delta\rho(r)V(|r-r'|)\,\delta\rho(r') + \frac{g}{2}\mu_B\psi^\dagger\vec{\sigma}\,\vec{B}\psi\,, \tag{29}$$

where ψ is now a two-component field of composite fermions with effective mass m. A_j is the vector potential for external magnetic field. Note that the Jain sequence for the fully polarized FQHS follows if $p\downarrow = 0$. A further specialization $p\uparrow = 1$ yields the Laughlin sequence. And finally, the choice $s = 0$ in eq. (28) gives the IQH states. Clearly, for the last case there is no CS term at the tree level.

The model proposed above is general enough to accommodate the known sequences of fully polarized systems. Let us consider the singlet states now. The singlet states correspond to $p\uparrow = p\downarrow \equiv p$, leading to the sequence $\nu = \frac{2p}{4sp+1}$. Thus the filling fractions $\nu = 2/5$, $2/3$ are singlet states corresponding to $p = \pm 1$, $s = 1$ respectively. Note that the sequence is exactly the same as that was obtained by Wu *et al* [31], and does indeed accommodate all the known experimentally observed states with $\nu < 1$ and also maintain consistency with the numerical result that all even numerator states are unpolarized. In the limit $p \to \infty$, $\nu \to 1/(2s) \Rightarrow$ that all even denominator states may also be unpolarized. Further by particle-hole symmetry, the states $2 - \nu$, and the states $2 + \nu$ which are obtained by the addition of LL [31], are also unpolarized. It is indeed true that the even-numerator levels such as $\nu = \frac{4}{3}, \frac{8}{5}$ and $\frac{10}{7}$ and [22] and even-denominator state like $\nu = \frac{5}{2}$ have been experimentally observed to be unpolarized [35].

Suppose $p\uparrow \neq p\downarrow$, which naturally leads to partially polarized states. The effective cyclotron frequency $\bar{\omega}_c$ is related to $\omega_c = eB/m^*$ by $\omega_c = \bar{\omega}_c[2s(p\uparrow + p\downarrow) + 1]$. There are two interesting limits that one can consider. The high and the low Zeeman energy limits. We have already considered the former above. Let us consider the latter.

For small Zeeman energies, we may take $p\uparrow = p\downarrow + 1 = p$ (say). Then

$$\frac{\Delta\rho}{\rho} = \frac{1}{2p-1}; \; \nu = \frac{2p-1}{2s(2p-1)+1}. \tag{30}$$

These states are indeed partially polarized, becoming fully polarized when $p = 1$. Then $\nu_{p=1} = 1/(2s+1)$ is again the Laughlin sequence known to be completely polarized [26]. It is interesting that in either of the limits (small or large) of Zeeman energy, the Laughlin sequence is fully polarized. The case $s = 1$ yields the sequence obtained by Wu *et al* [31]. Particle-hole symmetry and the addition of LL imply again that $2 - \nu$ and $2 + \nu$ are also spin-polarized. It turns out that the sequences given above exhaust almost all known integer and fractional states with full, partial or no polarization.

We shall now study the fluctuations about the MF background and derive an effective action, from which the (linear) response functions can be deduced.

C. One loop effect in FQHE :

The method has already been indicated in section 3, and the only modification here is an additional integration over the fluctuations in the CS gauge fields as well. The response is to be evaluated for a weak external electromagnetic probe, which couples to both the spins in the same fashion. However, it is advantageous to introduce two external probe fields, $A_\mu^{\uparrow,\downarrow}$, one which couples to the up electron, and the other to the down spin. The electromagnetic response can of course be easily read off from the more general response functions.

The integration over the fermionic fields yields an effective action involving the form factors $\Pi_{\mu\nu}$ as in the case of the integer quantum Hall effect. The form factors will be diagonal in the spin space. The next integration over the CS gauge field fluctuations yields the required effective action which in the spin space will not be diagonal any more.

Performing the integrations in eq. (29), we obtain the effective action for the external probes to be

$$S_{eff}\left[A_\mu^\uparrow, A_\mu^\downarrow\right] = \frac{1}{2}\int \frac{d^3q}{(2\pi)^3} A_\mu^r(q)\, K_{rr'}^{\mu\nu}(\omega, q)\, A_\nu^{r'}(-q), \tag{31}$$

where the indices $r, r' = \uparrow, \downarrow$. Here $K_{rr'}^{\mu\nu}(\omega, q)$ is the effective polarization tensor, which also contains the response functions.

We are interested in the electromagnetic response tensor $K^{\mu\nu}$ which is related to $K_{rr'}^{\mu\nu}$ through

$$K^{\mu\nu} = \sum K_{rr'}^{\mu\nu}. \tag{32}$$

Again, by virtue of translational, rotational and gauge invariances, we have the resolution

$$K^{\mu\nu} = K_0(q^2 g^{\mu\nu} - q^\mu q^\nu) + (K_2 - K_0)(q^2\delta^{ij} - q^i q^j)\delta^{\mu i}\,\delta^{\nu j} + iK_1\varepsilon^{\mu\nu\lambda}q_\lambda\ , \quad (33)$$

where K_0, K_1 and K_2 are functions of ω and q^2. Clearly, the Hall conductivity will be given by K_1, at $\omega = q^2 = 0$.

D. A brief digression – Kohn modes :

The density-density correlation function can then be evaluated in the limit $q^2 \to 0$ as

$$K^{00}(\omega, q^2) \equiv -K_0 q^2 = -\left(\frac{e^2\rho}{m^*}\right)\frac{q^2}{\omega^2 - \omega_c^2} + O\left((q^2)^2\right) \quad (34)$$

which is the same for both the cases, *i.e.*, unpolarized and polarized states.

Note that eq. (34) depends only on ω_c, the actual cyclotron frequency. Indeed, while magnetic invariance is broken due to attachment of fluxes to particles, for translationally invariant system, the magnetic symmetry must be restored. Therefore the centre of mass of the particles should move with the actual cyclotron frequency ω_c, in accordance with Kohn's theorem [36]. Thus while the form factors in (18-20) with a dependence on $\bar{\omega}_c$ seem to violate Kohn's theorem, we see that the fluctuations of the CS gauge fields in fact do restore it.

E. Hall conductivity :

Finally, let me exhibit the explicit expression for the Hall conductivity of the system :

$$\sigma_H \equiv K_1(0,0) = \frac{\left(\Pi_1^\uparrow(0,0) + \Pi_1^\downarrow(0,0)\right)\theta_+}{\left(\Pi_1^\uparrow(0,0) + \Pi_1^\downarrow(0,0)\right) + \theta_+}\ , \quad (35)$$

which on substitution yields

$$\sigma_H = \nu(e^2/2\pi), \quad (36)$$

We have thus verified that σ_H does indeed get quantized at the filling factors obtained from the MF ansatz.

F. Wave functions in CS theory :

One of the most studied objects in QHE are the many body wave functions, ever since Laughlin pioneered it in his seminal paper. There has been extensive work on their determination, see Ref. [16, 19, 17, 37, 38]. It is, therefore, interesting to examine whether the CS theory that we have at hand can correctly reproduce the wave functions determined numerically.

Fradkin [39] has developed a general method of determining the many body wave function in any given field theory. Strictly speaking, what one determines is the modulus squared, but if one knows before hand that the wave-function is a monomial, and that there are no gauge complications, the wave function can also be determined – after taking the root and projecting the antisymmetric / symmetric part depending on whether we are dealing with fermions or bosons.

As Lopez and Fradkin [40] have shown, for deriving the absolute square of the many body wave function for the ground state of a field theory, one needs the generating functional of equal-time correlation functions. Let us consider the density representation. The eigen states of the density operator, $\delta\hat{\rho}_r(X)$ provide a basis for (the subspace of) states with a fixed number of particles having a spin index r. The idea is to project out the ground state of the system in this basis.

Following Lopez and Fradkin [40], $|\psi|^2$ may be written as

$$\left|\psi[\rho_\uparrow, \rho_\downarrow]\right|^2 = \int [dA_0^\uparrow][dA_0^\downarrow] \mathcal{Z}[A_0^\uparrow, A_0^\downarrow] \exp\left[-ie \int \frac{d^2q}{(2\pi)^2} A_0^r(q) \delta\rho_r(-q)\right], \tag{37}$$

where $\delta\rho_r(q)$ is the Fourier transform of the density fluctuation

$$\delta\rho_r(X) = \sum_{i=1}^{N_r} \delta(X - X_i^r) - \rho_r, \tag{38}$$

which is the eigen value of $\delta\hat{\rho}_r(X)$. Here N_r is the number of particles with spin index $r\,(=\uparrow, \downarrow)$, ρ_r is the corresponding mean density, and $X_i(r)$ is the position of i-th particle.

Now the integrations over $A_0^\uparrow$ and $A_0^\downarrow$ in eq. (37) yield

$$\left|\psi[\rho_\uparrow, \rho_\downarrow]\right|^2 = \exp\left[\frac{e^2}{2} \int \frac{d^2q}{(2\pi)^2} \delta\rho_r(q) \kappa_{rr'}^{-1}(q) \delta\rho_{r'}(-q)\right]. \tag{39}$$

Let me exhibit the explicit forms for the singlet and the fully polarized states. I am interested only in the Jastrow forms. The Gaussian term can also be obtained by introducing a long distance cut off, much larger than the magnetic length and the interparticle separation Consider the singlets first. They correspond to $\rho_\uparrow = \rho_\downarrow = \rho/2$ and $N_\uparrow = N_\downarrow = N/2$ where N is the total number of particles. Consequently, $p_\uparrow = p_\downarrow \equiv p$. Then, as Mandal and Ravishankar have shown, the Jastrow part of the many body wave function for the ground state of unpolarized QHS having filling fraction $\nu = 2p/(4sp+1)$ is given by

$$\left|\psi(X_1^\uparrow, \ldots, X_{N/2}^\uparrow, X_1^\downarrow, \ldots, X_{N/2}^\downarrow)\right|^2 = \prod_{i<j}^{N/2} \left|X_i^\uparrow - X_j^\uparrow\right|^{2(2sp+1)/p}$$

$$\times \prod_{k<l}^{N/2} \left|X_k^\downarrow - X_l^\downarrow\right|^{2(2sp+1)/p} \prod_{i,k}^{N/2} \left|X_i^\uparrow - X_k^\downarrow\right|^{2(2s)}, \tag{40}$$

where $X_i^\uparrow\,(X_i^\downarrow)$ represents the co-ordinate of i-th spin-up (down) particle and $l = (eB)^{-1/2}$ is the magnetic length of the system.

Similarly for the fully polarized states,

$$|\psi|^2_{\nu = p_\uparrow/(2sp_\uparrow+1)} = \prod_{i<j}^{N} \left|X_i^\uparrow - X_j^\uparrow\right|^{2(2sp_\uparrow+1)/p_\uparrow} \tag{41}$$

which was first obtained by Lopez and Fradkin [21] in their pioneering work, considering spinless system from the very beginning. The $|\psi|^2$ for fully polarized Laughlin sequence can be obtained for $p_\uparrow = 1$ from eq. (41).

How good are the model wave functions ? This is an important question because we are, for the first time, probing a many body correlation. As the discerning reader may check, the above expression correctly reproduces the celebrated Laughlin wave functions for the fully polarized Laughlin sequence $\nu = 1/(2s+1)$. Take the example $\nu = 2/5$ in the singlet state, and the wave function proposed by Halperin [19] follows. The IQH state for $\nu = 2$ in the singlet state agrees with the one proposed earlier by Lee [41]. Finally, they are in consonance with the Jain prescription for constructing the FQH states from the corresponding IQH states, by multiplying with additional Jastrow factors. One may thus say that there is at last a rather strong evidence, albeit *a posteriori* for the correctness of the CS theory of QHS.

Not all is well, though, and let me dwell on the problems. This has a direct bearing on the confirmation that CFM has received in the past several years. Naturally, I am considering the even denominator states, and in particular, the $\nu = 1/2$ state. What do we get of the wave functions for these states ? The first problem is, of course, with the cut off which cannot be imposed since the magnetic length diverges. To add to that, the Jastrow forms in $|\psi|^2$ have exponents which are multiples of the form $4m$, where m is an integer. Clearly, the antisymmetric projection of the wave function vanishes identically. Unmistakably, CS theory has to be applied with abundant care for the even denominator QHS. At first sight it might appear even inapplicable. However, as I discuss in the next section, the remarkable work of Halperin *et. al.* [42] applies the CS theory for $\nu = 1/2$ quite successfully. It is nevertheless useful to remember the above remarks as the above work is also beset with some problems, which I will touch upon in the next section.

G Analytic properties of the wave functions

There is yet another aspect of the wave functions derived from the CS theory. As Lopez and Fradkin [21] observed in their first study, the IQH states are all non-analytic, since the Jastrow forms come with fractional exponents, and are therefore multiple valued. One is used to writing the Slater determinant for these states. So one has to understand what the above result means. In fact, most of the QHS, integer or fractional, are nonanalytic in the above sense. On the other hand, one expects the wave functions to have a support entirely in the basis provided by the lowest Landau level, as indeed prescribed by Jain (see eq 26). One has to understand the source of this multivaluedness.

It is possible that these non-analytic wave functions are simply wrong, coming from an effective theory which fails in these cases[4] ? If it were to be so, we are in a piquant situation, not having a controllable parameter that demarcates the reliable from the unreliable domain. Let me, therefore, proceed to elucidate the physical meaning of these exponents. Recall that in the powerful laughlin analysis [16] of the $1/(2s+1)$ states, the exponents of the Jastrow form, have in fact, the significance of relative orbital angular momentum. To gain a similar insight, let me rewrite the Jastrow froms as

[4] B. I. Halperin, private communication.

$$|\psi|^2 = \prod_{i<j}^{N/2} \left|X_i^{\uparrow} - X_j^{\uparrow}\right|^{2\left((1/\theta)+(1/\Pi_1^{\uparrow})\right)e^2/(2\pi)} \prod_{k<l}^{N/2} \left|X_k^{\downarrow} - X_l^{\downarrow}\right|^{2\left((1/\theta)+(1/\Pi_1^{\uparrow})\right)e^2/(2\pi)}$$

$$\times \prod_{i,k}^{N/2} \left|X_i^{\uparrow} - X_k^{\downarrow}\right|^{2(1/\theta)e^2/(2\pi)}, \tag{42}$$

where $\Pi_1^{\uparrow,\downarrow}$ are evaluated at $\omega = 0, q^2 = 0$.

From the above equation, it is clear that the exponent is the sum of two contributions, the strength of the CS term at the tree level, which always gives an integer phase, and the strength of the induced CS term Π_1. The latter contribution, which gets determined by the 1-loop correction, is most of the time non-analytic, depending on $p_{\uparrow}$ and $p_{\downarrow}$. Indeed, the exponents describe the number of effective vortices associated with a particle which is seen by others and therefore the exponents of the Jastrow form between like spin particles differ from the same between unlike spins. It is natural that Ψ should reflect the nature of vortices associated with a fermion. In other words, Ψ is determined from the density-density correlations which represent, in fact, the change in local density of the system and hence the change in CS magnetic field This causes a change in the local current which is represented by the vortices. These vortices, at long distances that matter to us, leave their signature as phases that accrue by the Aharanov-Bohm mechanism. Since the CS term is exact it would be very unlikely that there would be additional perturbative corrections of higher orders which can restore analyticity. It thus appears that the wave functions determined by the CS theory are realistic, barring the ones for even denominators.

H. The special regime : ν near $1/2s$:

I now address the special case of even denominators, and the states in their neighbourhood Recall that these states corresponding to very small values of the effective magnetic field, and very large values of the magnetic length and p. It is not straight forward to use the CS formalism for even denominator states. As we have seen, the wave functions turn out to be hopelessly wrong, as the antisymmetric projection vanishes identically. It is therefore not guaranteed that the response functions that are determined by employing RPA or some other approximation should be reliable. Therefore, this calls for care, and in an incisive analysis, Halperin *et. al.*[42] (HLR) studied the even denominator states within the CF theory. Let me summarize their main findings.

As a consequence of the Jain prescription, the ground state and the low-energy excitations for $\nu = 1/2$ state ought to be described by a modified Fermi-liquid theory bearing some resemblance to the theory of electrons in zero magnetic field. One important effect of the CS gauge field fluctuations is to make a large renormalization of the effective mass of CF. Halperin *et al* [42] (HLR) showed that for interaction of the Coulombic form $V(r) = e^2 \varepsilon r$, the effective mass diverges logarithmically at low energies as

$$m \sim \frac{(2s)^2}{2\pi}\left(\frac{\varepsilon \kappa_F}{e^2}\right)|\ln \omega|, \tag{43}$$

where ω is the energy with respect to Fermi energy. They also found that if the interaction falls off faster than $1/r$ at large distances, the low energy density fluctuation gets more effective, and as a result, m^* diverges more strongly than the logarithmic behaviour shown above. Thus the usual Landau theory does not hold. The crux of the HLR approach is that one can determine m^* either self consistently, or take the above equation as the defining expression for m^*. If one did that, HLR argue, then the energy gap for the Jain sequence would have the form

$$E_g = \frac{2^{\frac{1}{2}}\pi}{2s}\,\frac{e^2}{\varepsilon \ell_b}\,\frac{1}{D(\ln D + C)}, \tag{44}$$

where $D \equiv |2sp+1|$, with the constant C depending on the short distance behaviour of the interaction. The above expression was derived for large values of B.

The values of effective mass have been experimentally determined by several groups from the Shubnikov-de Haas oscillations observed in ρ_{xx} near $\nu = 1/2$, analogous to the electrons near zero magnetic field. If the standard theory of Shubnikov-de Haas oscillation is applied to the composite fermions, the value of m^* is found to be finite at $\nu = 1/2$ and is consistent with the value extracted from a measurement of the gap of activation. This is contrary to the findings of HLR. The drawback of the treatment of HLR for determining m^* is that the procedure is not gauge independent. However, the response functions that they found is gauge independent. Recently Chari *et al* [43, 44] have proposed a gauge-invariant way to determine m^* from the correlation functions. They find a finite m^* but it depends on the ultraviolet cut off momentum Λ which is the typical scale for the size of the flux tube attached with the composite fermions. In the limit $\Lambda \to \infty$, the result of HLR is recovered, and in the limit $\Lambda \to 0$, the mean field becomes exact. However, it appears that some experiments on Shubnikov-de Haas oscillations show a sharp rise in m^* close to $\nu = 1/2$, as predicted by HLR. Further work on both experiment and theory are necessary to resolve this issue. From the theoretical side, it is necessary to consider disorder in this problem as it will cause local density fluctuation.

Let us proceed further to discuss the response functions. Mere substituation of m^* for m in the random phase approximation (RPA) theory for evaluating correlation functions violates Kohn's theorem (which states that for a translationally invariant system, density-density correlation should have pole at cyclotron frequency $\omega = eB/m$ as shown in subsection D) and the f-sum rule :

$$\int \mathrm{Im}\, K^{00}(q,\omega)\,\omega\, d\omega = \frac{\pi\rho}{m} q \tag{45}$$

where K^{00} is the density-density correlation. Simon and Halperin [45] (see also Ref. [46]) later proposed a modification of the RPA, by introducing a Landau parameter of p-wave interaction channel which cancels the effect of m^* over m, to respect these requirements in a translationally invariant system.

Employing RPA (same as the 1-loop fluctuations we are studying) with the above improvements, HLR determine the diagonal conductivity σ_{xx} and find it to be $\sigma_{xx}(\bar{q}) = \frac{e^2}{4h}\frac{q}{\kappa_F}$, which implies a relaxation which $\sim q^3$ for short range potentials. The above expression for the diagonal conductivity is valid when $ql \gg \hbar$. In the opposite limit, $\sigma_{xx} = \frac{e^2}{4h\kappa_F l}$. Thus the resistivity tensor has the form $\rho_{ij} = \rho_{ij}^{(CS)} + \tilde{\rho}$, exhibiting the correction to the usual CS contribution. Most interestingly, they find that as one moves away from $\nu = 1/2$ by varying the magnetic field, $\sigma_{xx}(q)$ is not monotonic when $q\ell > 2$, predicting a series of maxima. This feature was the first to be seen experimentally.

7. Experimental verification of CFM – $\nu = \frac{1}{2}$

I first consider the surface acoustic wave experiments by Willet *et. al.* [47, 48], which can be understood in terms of the maxima in σ_{xx} predicted by the CS theory. Willett *et. al.* [47, 48] first observed some unusual features in the behaviour of surface acoustic wave (SAW) propagation near $\nu = 1/2$. The interaction between SAW and electrons occur through the piezoelectric effect in GaAs by means of the longitudinal electric field. The response of the system comes through the diagonal conductivity $\sigma_{xx}(\omega, q)$, where ω and q are the frequency and wave vector of the SAW. At $\nu = 1/2$, $\sigma_{xx}(\omega, q)$, is nonzero as the system is gapless. The finite conductivity leads to an attenuation of the amplitude of SAW ; and the velocity of SAW decreases with an increase in $\sigma_{xx}(q)$. Therefore, one expects that at given q, there should be a dip in the SAW velocity near $\nu = 1/2$ which was termed by Willet *et. al.* [47, 48] as "SAW anomaly". The HLR analysis predicts that the maxima occur whenever the cyclotron radius R_c and the wavelength $\lambda \equiv 2\pi / q$ are related by $R_c = (n + 1/4)\lambda$. Note that the maxima in the conductivity correspond to the minima in the SAW velocity. And experiments have in fact, seen a splitting in the SAW velocity minima, in agreement with the prediction of the CS theory, based on the CFM. These experiments constitute the first experimental evidence for CF.

There are other experiments that put the CF picture on a strong footing. (i) Du *et al* [49] find that the excitation gap, which is obtained from the activation in diagonal resistivity ρ_{xx}, is proportional to $\bar{B}$, in agreement with the prediction of the CFM. They further find that the effective mass m^* is independent of $\bar{B}$. Recall that on the other hand, HLR have argued that m^* diverges logarithmically for Coulomb interaction and that it has a power law divergence for other short range interactions as $\nu \to 1/2$, where CF form a Fermi liquid. (ii) Three experiments [50-52] have treated the oscillations in diagonal resistivity ρ_{xx} around $\nu = 1/2$ as Shubnikov-de Haas oscillations (SDHO) of CF, in analogy with SDHO of free electrons near $B = 0$. The effective mass of CF is then determined from SDHO. Though all these experiments are in favour of CF, they are mutually conflicting : while Leadley *et al* [50] have reported a finite mass m^* at $\bar{B} = 0$, and that m^* increases linearly with $|\bar{B}|$, Du *et al* [51] and Manoharan *et al* [52] have observed a 'drastic enhancement' of CF mass as $\nu \to 1/2$, indicating a novel Fermi liquid at $\nu = 1/2$. Although both Du *et al* [51] and Manoharan *et al* [52] have observed a diverging value for m^* as $\nu \to 1/2$, the former have obtained a much faster divergence compared to the latter (iv) Kang *et al* [53] have reported the existence of cyclotron motion of CF with radius R_c at $\nu = 1/2$ by their transport measurement in an antidot superlattice. (v) Goldman *et al* [54] have performed a transverse magnetic focussing experiment and observed quasiperiodic resistance peaks near $\nu = 1/2$. Moreover, they have observed that the quasiperiodic structure occurs on one side of $\nu = 1/2$, settling correctly the charge of the CF to be negative. They have also found that the charge carriers experience an effective magnetic field $\bar{B}$, in agreement with the CF picture.

(vi) Apart from the above transport measurements, there are plenty of data available in thermal measurements. Ying *et al* [55] have reported that thermopower measurements at $\nu = 1/2$ and 3/2 are consistent with the presence of a CF Fermi surface. Based on their data on thermopower. Bayot *et al* [56] have concluded that CF exhibit IQHE. (vii) There also exists a time-resolved magneto-luminescence experiment to study the hierarchy of FQHE states.

Kukushkin *et al* [57] have observed a striking symmetry in the dependence of chemical potential discontinuity on the filling factor for different families (different values of s) of the Jain sequence. They find a linear dependence of the chemical potential discontinuity on magnetic fields starting at $\nu = 1/2$, 1/4 and 1/6. (viii) Finally, Kukushkin *et al* [58] have studied the influence of disorder on the properties of 2DEG in the vicinity of $\nu = 1/2$. They have observed that ρ_u at $\nu = 1/2$ is very sensitive to the disorder level in 2DEG. This observation is in agreement with the theory of half filled Landau level in CF picture. It is noteworthy that the results are obtained for fully polarized samples.

Impressive that the above list is, let us remember that all the experiments are near the filling fraction 1/2, which is not very well understood theoretically. More pertinently, it is necessary to be able to identify observables / measurements that allow a determination of all the parameters that enter into the model. Since the response functions in the fully polarized phase are all dependent only on the actual filling fraction, the two ingredients, the effective number of filled Landau levels, p, and the number of flux tubes that get attached to the electron continue to be hidden. I now show how QHS with the spin degree of freedom allows us to determine these parameters unambiguously.

8. Direct verification of the CFM

This section may also be dubbed as 'direct determination of the CFM parameters'. The idea is to exploit the fact that the QHS can reside in more than one spin state by keeping the actual filling fraction fixed. I therefore, return to the study QHS with the spin degree of freedom and show how one can measure directly the parameters p, s [59-61].

The central idea is to observe that the correlation/response functions are now richer in their content. Indeed, consider the charge density correlation (CDC) and the spin density correlation (SDC). For a QHS is a fully polarized state, they are the same. If the system goes to a partially polarized state, with the value of n in tact, the SDC will change, while the CDC remains the same. For example, the state $\nu = 2/3$ is known to exist in the fully polarized as well as the singlet states.

The charge density correlation can be obtained from the lagrangian of the doublet model [60] as

$$K^{00}(\omega, q^2) \equiv \sum_{r,r'} K^{00}_{rr'}(\omega, q^2) = -q^2 \frac{(\Pi_0^{\uparrow} + \Pi_0^{\downarrow})\theta_+^2}{\mathcal{D}(\omega, q^2)}. \tag{46}$$

On the other hand, the spin density correlation is given by

$$\sum(\omega, q^2) = \sum_{r,r'} \left[K^{00}_{rr'} \delta_{rr'} - K^{00}_{rr'}(1 - \delta_{rr'}) \right]. \tag{47}$$

For unpolarized states, $\Pi_0^{\uparrow} = \Pi_0^{\downarrow} \equiv \Pi_0$. Thus Σ gets the simpler form,

$$\sum\nolimits_{unp}(\omega, q^2) = 2\Pi_0\,(\omega, q^2) q^2. \tag{48}$$

In the limit of low q^2, CDC and SDC are respectively given by

$$K^{00}(\omega, q^2) = -\left(\frac{e^2 q}{m^*}\right) \frac{1}{\omega^2 - \omega_c^2} q^2 + \mathcal{O}(q^4), \tag{49}$$

$$\sum(\omega, q^2) = -\left(\frac{e^2 q}{m^*}\right)\left[\frac{(p_\uparrow - p_\downarrow)^2}{(p_\uparrow + p_\downarrow)^2}\frac{1}{\omega^2 - \omega_c^2} + \frac{4p_\uparrow p_\downarrow}{(p_\uparrow + p_\downarrow)^2}\frac{1}{\omega^2 - \overline{\omega}_c^2}\right]q^2 + O(q^4)\ . \quad (50)$$

We see from Eq. (49) that CDC preserves the Kohn mode [36] of excitation. On the other hand, SDC in Eq. (50) shows a new mode of excitation at $\overline{\omega}_c$ apart from the actual cyclotron energy w_c. Interestingly, in the case of unpolarized QHS for which $p_\uparrow = p_\downarrow$, *only* the mode at $\overline{\omega}_c$ survives at $\boldsymbol{q} = 0$. This, in fact, gives the measure of energy scale for CF. Equations (49) and (50) are the same as the pure RPA result since we are in the regime of very low $\boldsymbol{q}$.

A. The spin transitions :

Let us consider SDC in the static limit. Then

$$\Sigma(0, \boldsymbol{q}^2) = \boldsymbol{q}^2\left(\frac{e^2 m^*}{4\pi^2\rho}\right)\left[\frac{(p_\uparrow - p_\downarrow)^2}{(p_\uparrow + p_\downarrow)^2}v^2 + 4p_\uparrow p_\downarrow\right]. \quad (51)$$

Experimentally, Eisenstein *et al* [25] and Engel *et al* [24] have observed spin transitions in QHS with filling fractions $v = 2/3$ and $3/5$. By an increase of Zeeman energy, QHS at $v = 2/3$ ($p_\uparrow = p_\downarrow = -1, s = 1$) and $v = 3/5$ ($p_\uparrow = -2,\ p_\downarrow = -1, s = 1$) undergo a spin transition from their respective phase of no polarization and partial polarization to fully polarized phase ($p_\downarrow = 0$) keeping $p_\uparrow + p_\downarrow$ fixed. It is clear that the effective number of LL filled by up (or down) spins $p_\uparrow$ ($p_\downarrow$) acts as an order parameter in the spin transitions. The value of Σ, accordingly, changes discontinuously in the spin transitions. Indeed, the ratio of the values of Σ between the unpolarized and fully polarized phases is given by $\Sigma_{unp} / \Sigma_p = 4p_\uparrow^2 / v^2$. Therefore, the ratio of $\Sigma(0, \boldsymbol{q}^2)$ in unpolarized and fully polarized phase would determine $p_\uparrow$ ($= p_\downarrow$) in the unpolarized phase unambiguously.

There is one significant observation here. The ratio does *not* depend on any parameter that is system specific – be it the mass, the charge or the density. We have seen that m^* has a dependence of B, the nature of which is not settled experimentally yet [50-52]. This independence holds even when the transition is to a partially polarized state, instead of the singlet state. In other words, we can now determine $p_\uparrow$ and $p_\downarrow$ unambiguously. The order parameter shows a discontinuity in all the spin transitions.

Next comes the measurability of the above quantities. It turns out the neutron scattering (quite difficult to perform because of the planar geometry !) can be used for this.

B. Neutron scattering :

In the standard neutron scattering experiments [62], (in this case, the scattering is in the plane of the sample), the differential scattering cross section is given by

$$\frac{d\sigma}{d\Omega} \propto \frac{\kappa_f}{\kappa_i}\left[S_c(q) + \frac{\sigma_\Sigma}{\sigma_c}S_\Sigma(q)\right] \quad (52)$$

where $\boldsymbol{k}_i$ and $\boldsymbol{k}_f$ are the momentum of the incident and scattered neutrons, $\boldsymbol{q} = \boldsymbol{k}_i - \boldsymbol{k}_f$ is the momentum transfer. In eq. (52), $S_c(q)$ and $S_\Sigma(q)$ are static charge and spin structure factors which are frequency integrated imaginary parts of the corresponding correlation functions. In eq. (52), σ_Σ/σ_c is the ratio of the spin and charge-dependent total cross sections.

In our case, the static structure factors are evaluated (from eqs. 49 and 50) as

$$S_c(q) = q^2\left(\frac{e^2}{2}\right)\nu\,, \tag{53}$$

$$S_\Sigma(q) = q^2\left(\frac{e^2}{2}\right)\left[\left|\frac{4p_\uparrow p_\downarrow}{p_\uparrow + p_\downarrow}\right| + \frac{(p_\uparrow - p_\downarrow)^2}{(p_\uparrow + p_\downarrow)^2}\nu\right]. \tag{54}$$

Note that unlike the parent expressions in eqs. (49 and 50), the above expressions are free from any dependence on ρ, and m^* which by now is known to possess a dependence on the magnetic field [50-52]. As expected, $S_c(q)$ is proportional to ν irrespective of the spin phase. The form of $S_\Sigma(q)$ depends on the spin phase, however. In the fully polarized phase, $S_c(q) \equiv S_\Sigma(q)$. In the unpolarized phase, $S_\Sigma(q) \propto p_\uparrow$ while, in general, for a partially polarized phase, it depends on both $p_\uparrow$ and ν.

What is then required is a determination of $S_\Sigma(q)$ in unpolarized or partially polarized phases which may be accomplished by two different ways : (i) By a measurement of neutron scattering cross section in the fully polarized phase, which yields $S_c(q)$, followed by the same in the partially polarized (it could be a singlet phase as well) from which $S_\Sigma(q)$ can be determined. Since the structure factors are independent of the particle density ρ, their determination is unambiguous, no matter how the different spin phases are obtained — be it by changing the particle density or by a tilting angle experiment. (ii) The second method corresponds to the experiments entirely in the unpolarized or partially polarized phase, whichever is the relevant one. However, here one needs two probes —X-ray and neutron. X-ray scattering experiment will determine $S_c(q)$ and then the neutron scattering experiment can be used to determine $S_\Sigma(q)$, with the knowledge of $S_c(q)$.

Thus in both the unpolarized and partially polarized phases, $S_\Sigma(q)$ determines the composite fermion parameter $p_\uparrow$ which is identified as an order parameter in the spin transition, and can be measured experimentally by neutron scattering experiments. Therefore, neutron scattering experiment provides a *direct unambiguous* test of CF. The accuracy of eqs. (53 and 54) lie on the region of small angle scattering as it is valid only for low q^2.

There are two problems here. As already mentioned, neutron scattering experiments are notoriously difficult to perform. Also, a complete determination of all the CFM parameters is what the goal is. And as the gentle reader can see, there is nowhere any reference to the number of flux tubes that generate the CF. The Jain relation is thus still not verified completely.

The solution to this hurdle lies in determining the charge density and the spin density excitations, instead of just the static correlations. This determination (see [60]), has to be done by going beyond the RPA, within the so called time dependent Hartree-Fock approximation which incorporates the interaction between the quasi particle and the quasi hole (excitonic interaction). These collective excitations can then be probed by Raman scattering experiments.

C. Raman scattering :

By polarized and depolarized Raman scattering experiments, the modes of CDE and SDE (discussed in the previous subsection) can respectively be found out. The Raman intensity $I(\omega)$ is proportional to the imaginary part of the corresponding correlation functions which are known as spectral functions [63]. We are interested in that geometry of Raman scattering where the contributions of CDE and SDE in the cross sections get separated out. It corresponds to $\hat{e}_i \parallel \hat{e}_s$ for CDE and $\hat{e}_i \perp \hat{e}_s$ for SDE, where $\hat{e}_i$ and $\hat{e}_s$ are the direction of the polarization of the incident and scattered beam respectively. The respective separated cross sections are given by [63]

$$\frac{d^2\sigma_c}{d\Omega\, d\omega} \propto (\hat{e}_i \cdot \hat{e}_s)^2\, S_c(\omega, q); \tag{55}$$

$$\frac{d^2\sigma_\Sigma}{d\Omega\, d\omega} \propto (e_i \times \hat{e}_s)^2\, S_\Sigma(\omega, q) \tag{56}$$

Here $S_c(\omega, q)$ and $S_\Sigma(\omega, q)$ are the spectral functions for CDE and SDE respectively. The geometry in which only CDE are determined is known as *polarized* Raman scattering geometry, while SDE are determined in *depolarized* Raman scattering geometry.

In the limit $q^2 l_0^2 << 1$, most of the weight of CDC is in the plasmon mode *i.e.*, at ω_c for both fully polarized and unpolarized phase. Consider now depolarized Raman scattering experiment. In the *fully polarized* phase, it creates spectra very similar to the one in the polarized Raman scattering experiment because CDE and SDE are same in this phase. On the other hand, in the *unpolarized* phase, the highest intensity will be observed for the mode whose energy gap is $\bar{\omega}_c$, and would signal the existence of composite fermions. The above statements hold for a QHS which goes to the singlet state. For partially polarized states, such as $\nu = 3/5$, there will be three highest intensity peaks (in the same order of q^2) corresponding to one mode at ω_c and two modes at $\bar{\omega}_c$. In any case, one does have a handle on the CF since one is probing the $\bar{\omega}_c$ directly. Of course, the determination is not that clean as neutron scattering since we need to know the value of m^*. On the other hand, and this is a strong point in favour of these observables, the measurements are not tied to filling fractions equal to or near 1/2. Quite the contrary, since the whole approach of studying the CF fluctuations is reliable only if we stay away from that extreme region.

There is yet another fall out of the study of these collective modes, of the spin excitations as well as the spin waves. These can be employed, albeit within an approximation scheme, to understand and characterize the skyrmionic excitations in QHS. These excitations possess a topological order. And as I show below, they unveil the last of the hidden parameters – the number of flux tubes that attach to the electron !

9. Quantum Hall skyrmions

The spin degree of freedom is crucial here, and we deal with QHS in the so called vanishing Zeeman splitting limit. Let us ask what the effective action would be if we integrated the CS lagrangian over all the variables save the spin degree. Building upon the earlier work of Lee and Kane [41] and Stone [64], Sondhi *et. al.* [65] first studied this problem for the fully polarized Laughlin states. They argued that the effective action would be a non-linear $O(3)$ σ model in the spin variables, which is known to support a ferromagnetic ground state, even as the Zeeman splitting is switched off. We can follow Sondhi *et. al.* [65] in arguing that the system can produce collective excitations with non-trivial topological order ; these are identified to be skyrmions following the standard results available on the non-linear $O(3)$ σ model [66]. One cannot perform the path integral exactly, and so the value of the spin stiffness will have to be obtained from some other analysis such as the TDHFA described above. By such an analysis, Sondhi *et. al.* [65] showed that the topological charge density $q(r)$ of the skyrmions is proportional to the particle density fluctuation and is given by $\delta\rho(r) = \nu q(r)$, with $q(r) = \varepsilon^{ij}\hat{n}\cdot(\partial_i\hat{n}\times\partial_j\hat{n})/8\pi$. The topological charge $Q = \int d^2 r q(r)$ is an integer : positive for skyrmions and negative for antiskyrmions. The magnitude of Q represents how many times the spin configuration wraps around a unit sphere, and its sign, the mode of wrapping. For vanishing Zeeman energy, the skyrmions [66] are infinite and their energy comes only through the spin stiffness. While the interaction between skyrmions favours macroscopic skyrmions, the Zeeman term reduces their size.

This is not to say that the skyrmions are always the lowest energy charged excitations. For the Laughlin states, they are indeed so. Recently, Lilliehook *et al.* [67] have reported that they can further characterize them by specifying the charge to be $Q = 2$.

The natural question is regarding the nature of skyrmionic excitations for partially polarized states. For example, Jain and Wu [68] report that for the partially polarized $\nu = 3, 5$ integer states, the skyrmions are not the lowest energy excitations. This conclusion is based on a numerical analysis. Let me here indicate how the doublet model throws light on these topological excitations in a unified fashion.

At this point, I should mention that experiments [69] at $\nu = 1$ do detect charged quasiparticle excitations with macroscopic size and large spin, verifying the above predictions [41, 65, 70]. Subsequently, the presence of similar topological excitations called 'merons' has been suggested [71] for describing a novel phase transition observed in double layer systems [72].

Here I discuss only single layered systems, and discuss the relation between the topological and the electric charges, the charge and statistics of the quasiparticles, estimates for the spin stiffness, and finally the energetics.

It is convenient to introduce auxiliary CS fields and transform the fermions to composite bosons (CB). To accomplish this, we introduce yet another doublet of CS gauge fields, (in the same spirit of Ref. [59]). with the coupling matrix now given by

$$\Theta = \begin{matrix} \theta_1 & \theta_2 \\ \theta_2 & \theta_1 \end{matrix} \tag{57}$$

The conditions on the coupling can be easily stated. If $\theta_\pm$ be the eigen values, then the requirement that the resultant state should be fermionic imposes the condition $S_+ + S_- = 1$, where I have, as usual, parametrized $\theta_\pm = e^2/2\pi s_\pm$.

In this CB representation, let us resolve ψ as, $\psi = \sqrt{\rho}\phi\chi$ such that $\phi^*\phi = 1 = \chi^\dagger\chi$. Here χ is the CP_1 field which is related to the unit spin vector via $n^a = \chi^\dagger\sigma^a\chi$. A Bose condensation of the composite bosons leads to quantum Hall state in the original electronic system [73]. In other words, the net mean field felt by the composite bosons is zero. Therefore, the mean field configuration is taken to be $\langle B_\phi + B_\chi + B + b + b^+ + \tilde{b}\rangle = 0$, where

$\tilde{b} = \varepsilon^{ij}\partial_i(n_3 a_{\bar{j}})$ and B's and b's are the magnetic fields corresponding to respective gauge fields. The individual mean electric fields are also taken to be vanishing.

A word of caution here. The transformation that takes CF to CB is not unique ; either $s_\pm = 1/2$, implying that one flux quantum is seen by all the bosons, or $s_+ = 1, s_- = 0$ which would lead to a flux quantum being seen only by like bosons[5]. There are occasions where the choice matters. Let me denote the choices as I and II respectively for a future reference.

We are interested in the static properties of the skyrmions. The corresponding lagrangian is

$$\mathcal{L}[\chi, A\kappa] = \rho(i_\chi{}^\dagger\partial_0\chi) - \frac{K}{8}(D_j^{ab}n_b)^2 + \frac{1}{2}g\,\mu_B\bar{\rho}\bar{n}_3 B$$

$$-\frac{K}{2}\int d^2r\rho_t(r)\ln|r-r'|\,\rho_t(r')$$

$$-\frac{1}{2}\int dr'\,\delta\rho(r)\,V(|r-r'|)\,\delta\rho(r'), \tag{58}$$

where

$$\rho_t(r) = \frac{1}{2\pi e}\varepsilon^{kl}\partial_l\left[i\chi^\dagger\partial_\kappa\chi - \frac{e}{2\pi}\left(\frac{1}{\theta} + \frac{1}{\theta_+} + \bar{n}_3\frac{1}{\theta_-}\right)A_\kappa\right], \tag{59}$$

in writing which I have employed the notation $\bar{n}_3$ for the mean density, and also introduced an auxiliary gauge field A_λ defined by

$$eJ^\mu = \frac{1}{2\pi}\varepsilon^{\mu\nu\lambda}\partial_\nu A_\lambda\,, \tag{60}$$

which implements the current conservation.

Let us now demand that the solutions possess a finite energy. We immediately obtain $q(r) \equiv \varepsilon^{ij}\hat{n}\cdot(\partial_i\hat{n}\times\partial_j\hat{n})/8\pi$ asymptotically is

$$q(r) = \frac{e^2}{2\pi}\left(\frac{1}{\theta} + \frac{1}{\theta_+} + \bar{n}_3\frac{1}{\theta_-}\right)\delta\rho(r)$$

$$= (2s + s_+ + \bar{n}_3 s_-)\delta\rho(r), \tag{61}$$

which displays the relation between topological charge density and particle number density fluctuation. The above relation is of course valid only asymptotically, but if we assume that the configuration is smoothly varying, we can take the relation to hold everywhere. Note that it depends both on the polarization and the number of (different kind of) vortices attached to each boson, in general. For partially polarized QHS, the relation does not depend[6] on the filling fraction ν.

[5]other choices would not lead to CB, but cause anyonic phase between dissimilar spins.

[6]This is in disagreement with the assumption of Wu and Sondhi [74]. However, the relation is the same as $\delta\rho(r) = \nu q(r)$ for fully polarized $\nu = 1/(2s+1)$ states as is obtained by Sondhi *et al.* [65]

The charge of the quasiparticle, *i.e.*, skyrmions of topological charge $Q = 1$ is then given by $e^{*} = e/(2s + s_{+} + \bar{n}_3 s_{-})$, which we will come back to below. Similarly, the electronic charge of the skyrmions, having a topological charge $Q = m$, given by the configuration (64) below is $e/(2s + s_{+} + \bar{n}_3 s_{-})$.[7]

Now that we have fixed the topological charge, and the electric charge of the skyrmions[8], let us turn to the statistics obeyed by the skyrmionic excitations. That may be read off from the Hopf term in the lagrangian,

$$\mathcal{L}_H = \frac{e^{*2}(2s + s_{+} + \bar{n}_3 s_{-})}{2\pi} A_{0\chi} B_{\chi} \tag{62}$$

The statistics of the skyrmions is found to be $\pi/(2s + s_{+} + \bar{n}_3 s_{-})$. In the case of partially polarized quantum Hall states, the statistical phase depends on the polarization for the choice I of bosonization ; the statistics depends on the choice of the parameters $s_{\pm}$. However, for fully polarized Laughlin $1/(2s + 1)$ states, either choice of $s_{\pm}$ gives the same statistics $\pi/(2s + 1)$, in agreement with previous findings [76, 77] but in disagreement with Ref. [78] where the statistics is claimed to be dependent on the spin of skyrmions. Note that this quantum statistics is valid in the long wavelength limit only.

Finally, let me also exhibit the static rotationally symmetric spin vortex solutions :

$$\chi(\vec{r}) = \begin{pmatrix} \cos\frac{\alpha(r)}{2} \\ \sin\frac{\alpha(r)}{2} e^{im\phi} \end{pmatrix}, \vec{a}^{-} = a(r)\,\hat{e}_{\phi}\,. \tag{63}$$

Then the unit spin vector

$$\hat{n} = [\sin\alpha(r)\cos(m\phi), \sin\alpha(r)\sin(m\phi), \cos\alpha(r)]. \tag{64}$$

Given this configuration, the finite energy requirement holds if, asymptotically $(D_k^{ab} n_b)^2 = 0$. More explicitly, $a(r) \to (-m)/2r$ and $\cos[\alpha(r)] = \bar{n}_3$ as $r \to \infty$. If one opts for choice II, $a(r)$ does not exist at all.

The resultant configurations of the skyrmions are the following.

(i) *Fully polarized states* : The physical space is compactified to a sphere of unit radius such that $r = 0$ corresponds to south pole ($\alpha(0) = \pi$) and $r = \infty$ becomes north pole ($\alpha(\infty) = 0$). The spin at the north pole will be up while it is down at the south pole. The unit sphere of spins wraps the physical unit sphere exactly once, for skyrmions of topological charge $Q = \pm 1$.

[7]This disagrees with the result of Ref. [75] for the following reasons. The topological charge of the skyrmions depends on the relative phase between the components of the CP_1 field (see eq (63) below). Hansson *et al* [75] instead consider the absolute phases of the two components of the CP_1 field. In other words, they consider the charged vortex as well, apart from spin vortex Secondly, the model they use for partially polarized states is also not quite the same as what we have here. They consider two component CS gauge fields, with one component coupling to the charge density while the other couples to the spin density. In that case, the coupling matrix should have been diagonal as we have the coupling parameters $\theta_{\pm}$ in eq (57). They instead propose a nondiagonal coupling matrix which leads to a different (and incorrect) result.

[8]The reader will observe that the electric charge of the skyrmions in the Laughlin states is same as the fractional charge that he predicted for the excitations.

(ii) *Partially polarized states :* The skyrmions are not the usual ones in this case. Since $\cos\alpha(\infty) = \bar{n}_3 < 1$, $r = \infty$ does not become north pole, but corresponds to an angle $\cos^{-1}(\bar{n}_3)$. The unit sphere of spins wraps exactly once inside the physical sphere for $Q \pm 1$ in this case as well. The partially

(iii) *Unpolarized states :* The skyrmionic excitations are not possible in this case.

The second term in eq. (58) can be expressed [64] as $(\bar{\rho}/2)\vec{\mathcal{A}}\cdot\partial_0\hat{n}$ where $\vec{\mathcal{A}}(\hat{n})$ is the vector potential for a unit monopole. This term describes precession of spins in the tilted magnetic field. The effective Lagrangian for the long wavelength static skyrmions now becomes

$$\mathcal{L}[n] = \mathcal{L}_H + \frac{\bar{\rho}}{2}\bar{A}\cdot\partial_0\hat{n} - \frac{\rho_s}{2}(\nabla\hat{n})^2 + \frac{g}{2}\mu_B\bar{\rho}\bar{n}_3 B$$

$$- \frac{1}{2(2s + s_+ + \bar{n}_3 s_-)^2}\int d^2r' q(r)V(r-r')\,q(r'), \tag{65}$$

ρ_s is the spin stiffness which is normalized due to Coulomb interaction. The second and third terms in (65) is standard for a ferromagnetic system. The next two terms break the scale invariance The size and energy of the skyrmions depend crucially on the interplay between these two terms. As the value of the parameter $\tilde{g} = g\mu_B B/(e^2\varepsilon l) \to 0$, the size of the skyrmions increases while their energy decreases [65]. AT $\tilde{g} = 0$, the skyrmions are of infinite extent having the energy $4\pi\rho_s$.

The value of ρ_s can be estimated by comparing dispersion relation from the above Lagrangian [65] with the neutral long wavelength spin wave dispersion relation. The coefficient of the gradient term in the spin wave dispersion relation [79] is $\kappa = \frac{1}{4}\sqrt{\frac{\pi}{2}}\frac{e^2}{\varepsilon l_0}$ for fully polarized $\nu = 1/(2s+1)$ states and $\kappa = \frac{7}{16}\sqrt{\frac{\pi}{2}}\frac{e^2}{\varepsilon l_0}$ for partially polarized $3/(6s \pm 1)$ states. Here l_0 is the effective magnetic length which is related with actual magnetic length *via* $(l_0/l)^2 = p/\nu$, where $p = p_\uparrow + p_\downarrow$ is the effective number of Landau levels filled by CF.

(i) Determination of the parameter S

It remains to show how the parameters s can be determined by studing the skyrmionic excitations That would be possible if the as yet free parameter, the spin stiffness $\rho_s = \kappa/4\pi$ can be fixed. As I mentioned earlier, it is necessary to go beyond RPA for this, and an estimate from the so called TDHFA [79] yields, for the Laughlin states,

$$\rho_s = \frac{\sqrt{\nu}}{(2s+1)^2}\frac{1}{16\pi}\sqrt{\frac{\pi}{2}}\frac{e^2}{\varepsilon l}. \tag{66}$$

It should be emphasized that this result is obtained from the doublet model after incorporating the composite fermion requirement, and by using the time dependent Hartree-Fock approximation. Sondhi *et. al.* [65] estimated spin stiffness for $\nu = 1/3$ and $1/5$ states using the results of single mode approximation obtained by Girvin, Macdonald and Platzmann [80]. They obtained the value of $4\pi\rho_s = 0.024$ and 0.006 (in the unit of $e^2/\varepsilon l$) for $\nu = 1/3$ and $1/5$ states respectively. These numbers are in good agreement with that of Ref. [79] as 0.0201 and 0.0056 respectively obtained from eq. [66].

For partially polarized states, the topological charge depends on the filling fraction and the polarization as well if we employ choice I for bosonization. It is independent of the polarization for the choice II. For these states, the ambiguity in the choice of bosonization can, in principle, be resolved through the experiments that determine the excitation gap for skyrmions in the limit $\tilde{g} \to 0$.

Let us estimate the spin stiffness for $\nu = 3/(6s \pm 1)$. It is given by

$$\rho_s = \begin{cases} \dfrac{\sqrt{\nu}}{\sqrt{3}} \dfrac{1}{(2s+2/3)^2} \dfrac{7}{64\pi} \sqrt{\dfrac{\pi}{2}} \dfrac{e^2}{\varepsilon l} & \text{for } s_+ = \dfrac{1}{2},\ s_- = \dfrac{1}{2} \\ \dfrac{\sqrt{\nu}}{\sqrt{3}} \dfrac{1}{(2s+1)^2} \dfrac{7}{64\pi} \sqrt{\dfrac{\pi}{2}} \dfrac{e^2}{\varepsilon l} & \text{for } s_+ = 1,\ s_- = 0 \end{cases} \tag{67}$$

The determination of the spin stiffness has an importance beyond fixing its numerical values, in the VZS limit. As can be seen from the above expressions, the as yet undetermined parameter $2s$ of CFM gets determined thereby ! It is indeed remarkable that the attachment of flux which has a topological significance, could be probed directly by looking at the topological excitations, *viz.* the skyrmions. These experiments, if performed, would constitute a definitive confirmation of CFM–by virtue of determining all the CF parameters independently.

(ii) The energetics

Let us now consider briefly the energetics of the excitations. For fully polarized $\nu = 1/(2s+1)$ states, the energy required to create a fully polarized quasiparticle quasihole pair [74, 79], $(\sqrt{\pi\nu/2})(e^2/\varepsilon l)$, is greater than that for creating a skyrmion antiskyrmion pair of topological charge $Q = \pm 1$ at $\tilde{g} = 0$, which is obtained from eq. (66) as $(\sqrt{\pi\nu/2})(1/2\,(2s+1)^2)(e^2/\varepsilon l)$. The ratio of the energies for these two kinds of charged excitations is given by $E_{sk-ask} / E_{qp-qh} = 1/2\,(2s+1)^2$. Therefore, the skyrmions are the lowest energy charged excitations in the ground state of fully polarized states.

For partially polarized $\nu = 3/(6s \pm 1)$ states, the energy of fully polarized quasihole quasiparticle pair [79], $(3/4)(\sqrt{\pi\nu/6})(e^2/\varepsilon l)$ and the energy of skyrmion antiskyrmion pair, $8\pi\rho_s$, which can be determined from eq. (67). The ratio of these two energies is given by $E_{sk-ask} / E_{qp-qh} = 7/6\,(2s+2/3)^2$ and $7/6(2s+1)^2$ for choices I and II respectively. Thus, the skyrmions are not the lowest energy charged excitations in partially polarized integer states ($s = 0$) ; in agreement with Wu and Sondhi [74], although there is a possibility of such excitations. However, for both the choices, the skyrmions are, in fact, the lowest energy charged excitations in the partially polarized fractional states, contrary to the speculation of Wu and Sondhi [74]. We have pointed out that the prescription for obtaining composite bosons are not unique. The ambiguity has no impact on the Laughlin states, but does matter for partially polarized states. For these states, the ambiguity of the choices may be indeed resolved by experiments by determining the excitation gap for skyrmions in the limit $\tilde{g} \to 0$.

10. Alternative approaches and new developments

This short section merely mentions with, if at all, very brief elaboration on alternative and more recent approaches to the CS theory of FQHE.

First of all, mention must be made of CS bosonic approach to study FQHE[9]. Girvin and Macdonald [81] mapped the CF into CB by a further gauge transformation that attaches additional fluxes so as to transmute the statistics. At the MF level, the CB experience net zero magnetic field, and Girvin and Macdonald [81] showed that the transformed density matrix exhibits an algebraic odd diagonal long range order (ODLRO) which indicates that the hard bosons (are) superconducting. Read [82] also found a related but distinct ODLRO, which was determined numerically by Rezayi and Haldane [83] to be nonvanishing for $\nu = 1/3$, $2/5$. Zhang *et. al.*, [73] developed a CS Landau Ginzburg theory, using which Zhang [84] has derived the Laughlin wave functions and also the ODLRO of Girvin and Macdonald. Lee *et. al.* [85] have then studied the transitions between the QHS using the above theory, and Kivelson *et. al.* [91] have constructing the global phase diagram for QHS in the magnetic field – impurity plane.

A very interesting attempt to incorporate the Jain prescription for the CF and their wave functions, at the MF level itself, has been made by Rajaraman and Sondhi [86], by treating the gauge potential to be complex. The Laughlin and the Jain wave functions are derived without having to study the fluctuations that seem required otherwise. The non-hermitian hamiltonian is more complicated, and one wonders what the effect of fluctuations on the MF results would be. Even so, the approach has been successfully employed to study solitonic solutions in FQHE [87,88]. It is certainly worthwhile studying the quantum corrections to the MF results in this formalism.

Finally, there is a recent work of Shankar and Murthy [89, 90] who derive the Jain wave functions at the tree level itself, without having to introduce the complex gauge fields. The accomplish this by introducing into the n particle Hilbert space n additional oscillators, and n constraints. As an important application, they study the $\nu = 1/2$ state, and are able to determine the wave function, and the substantially renormalized charge and mass of the CF. This analysis is made, as much as all others, for low values of q and within RPA. And how and why RPA works is yet to be understood.

11. Summary

In summary, I have discussed in this article an effective theory for describing the Quantum Hall effects, in terms of a CS action. We saw that for 2DEG in an external magnetic field, the emergence of the CS action is natural (and unavoidable) at the 1-loop level. More significantly, it was shown that the introduction of the CS action at the tree level itself allows for an elegant expression of the CFM in the language of an effective field theory. The theory succeeds in capturing essential features of QHS that make them fascinating objects of study – the topological nature of very precisely quantized Hall conductivity. I further elaborated upon the very interesting region at and near $\nu = 1/2$ and showed how a careful treatment of the CS lagrangian sheds much light on the nature of CF. Finally, by considering QHS with spins not frozen, it was shown how *all* the parameters that enter into the CFM can be determined — independently, and unambiguously, provided one also looks at the skyrmionic excitations in the partially as well as the fully polarized states. Alternative and more recent developments are also alluded to very briefly in the previous section. In short, one may safely say that the CS interaction provides us with a convenient, reasonably successful and an elegant method for studying and understanding the enigma of Quantum Hall Effect.

[9]We employed it with profit in discussing skyrmionic excitations in the previous section.

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